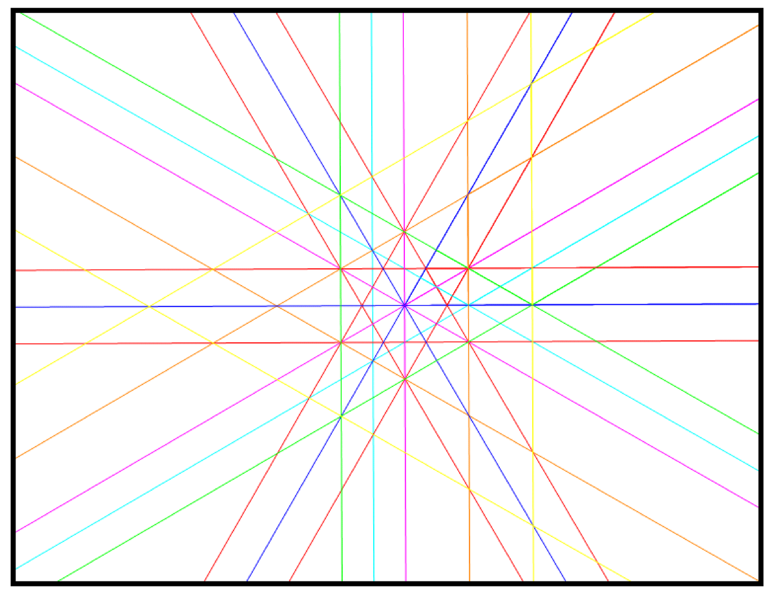
Highly Symmetric Quadri-Triangular Pseudoline Arrangements

Classifying quadri-triangular arrangements using the Kaleidoscopic beam technique

Samantha MacNeith, Dr. Leah Berman

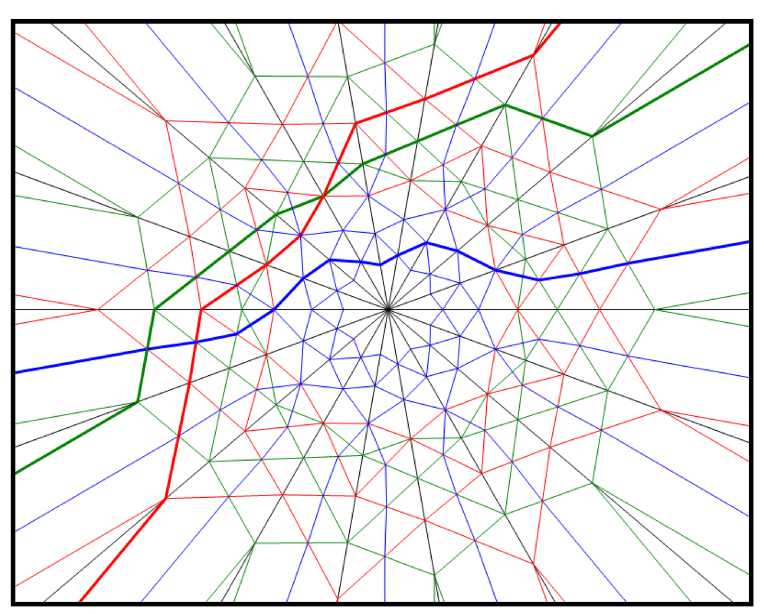
Linear Arrangements

An arrangement of lines is a finite family of lines which partition the plane into regions.[1]



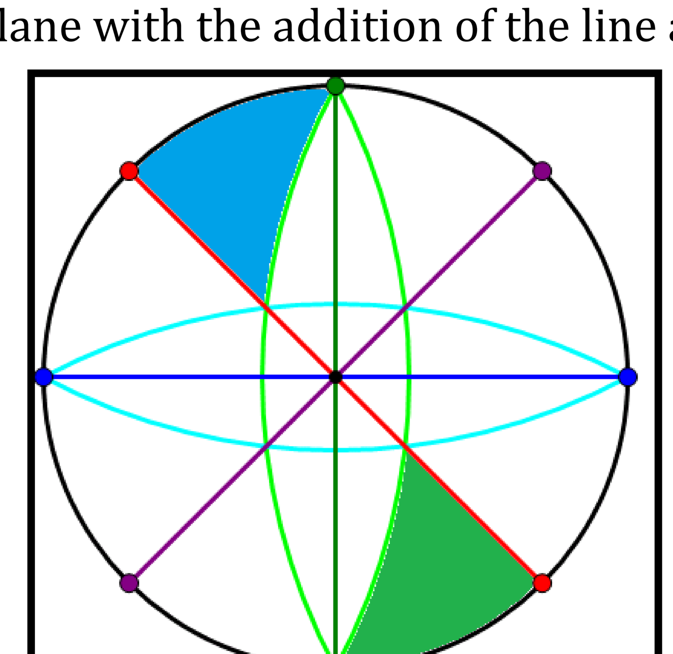
Pseudolinear Arrangements

An arrangement of pseudolines in E⁺ is a collection of non-self intersecting curves such that they are topologically equivalent to lines and only differ from lines in a finite piece-wise linear portion.[1] No pair of pseudolines may intersect more than once.



Extended Euclidean Plane

The Extended Euclidean Plane (E⁺) is analogous to the Euclidean plane with the addition of the line at Infinity.

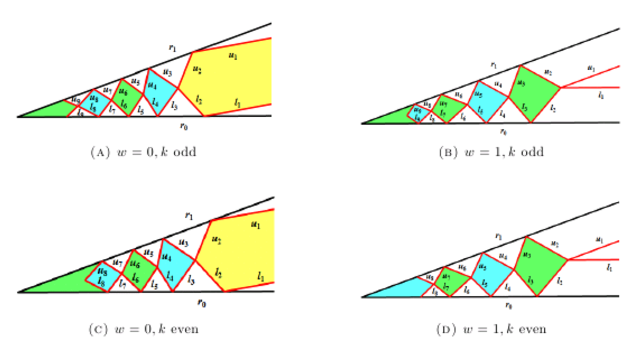


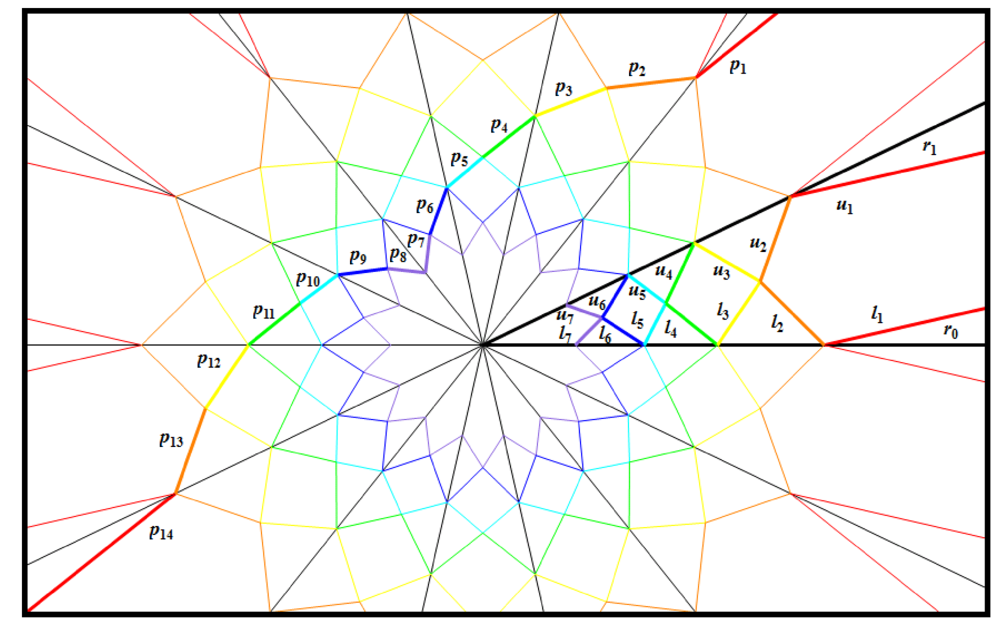
The line at infinity is the collection of the points at infinity. Every set of parallel lines meet at the same point at infinity. [1]

Bow-Tie Construction

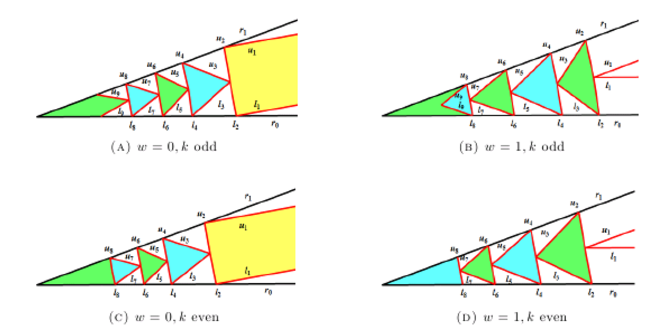
To construct a bow-tie arrangement with the symmetries of a regular k-gon, begin with a single beam pseudoline arrangement on 2k, then remove the angle bisector the generating wedge. Now reflect the generating wedge around the mirrors of the arrangement. The resulting arrangement contains only triangles and quadrilaterals.

Bow-tie arrangements are not simplicial in general.





Changing the angle of incidence can cause fewer quadrilaterals in the resulting arrangement.



Theorem 1: A single beam bow-tie configuration with the symmetries of a regular k-gon constructed using the Kaleidoscopic beam technique contains in it's generating wedge

w=0, even k: $k \leq p_3 \leq \frac{3k}{2}$ and $0 \leq p_4 \leq \frac{k+1}{2} + 1$

w=0, odd k: $k+1 \leq p_3 \leq \frac{3k+2}{2}$ and $1 \leq p_4 \leq \frac{k+1}{2}$

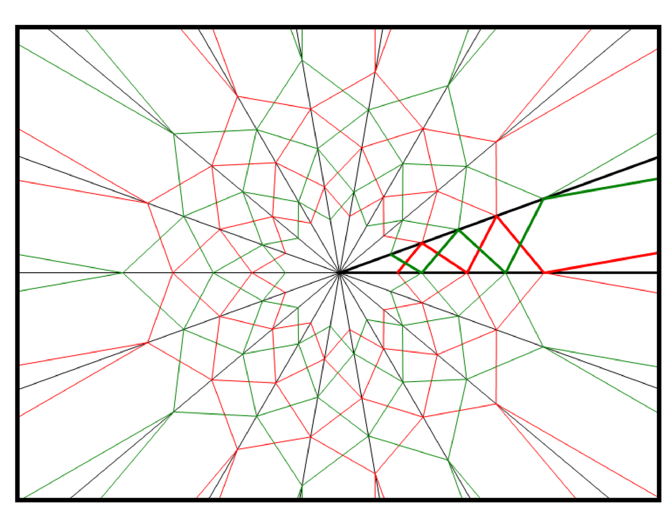
w=1, even k: $k+1 \leq p_3 \leq \frac{3k+1}{2}$ and $0 \leq p_4 \leq \frac{k}{2}$

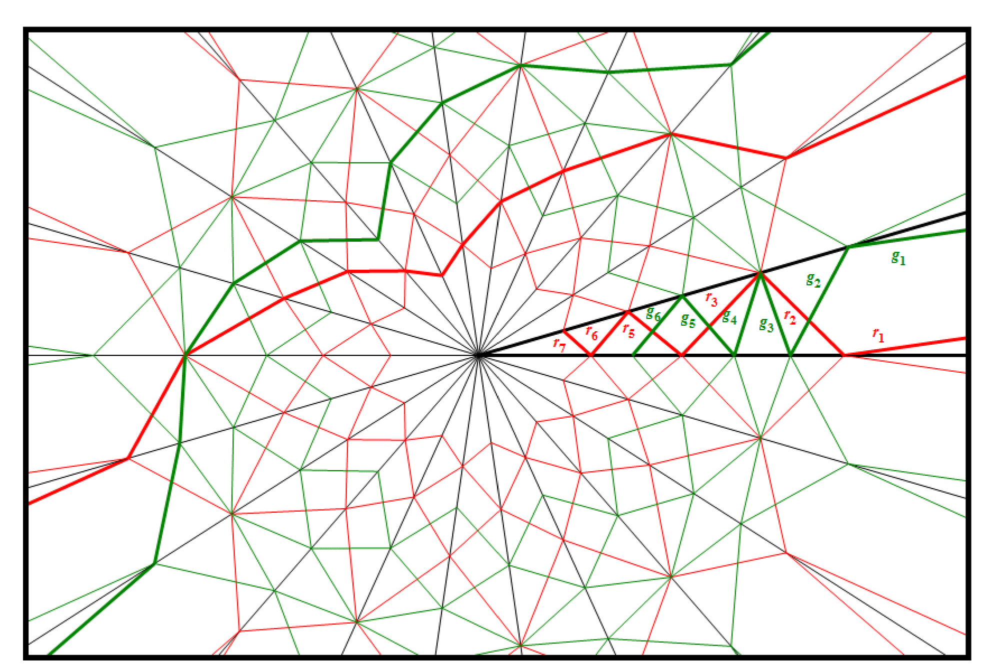
w=1, odd k: $k \leq p_3 \leq \frac{3k-1}{2}$ and $1 \leq p_4 \leq \frac{k+1}{2}$

Two Beam Quadri-Triangular Construction

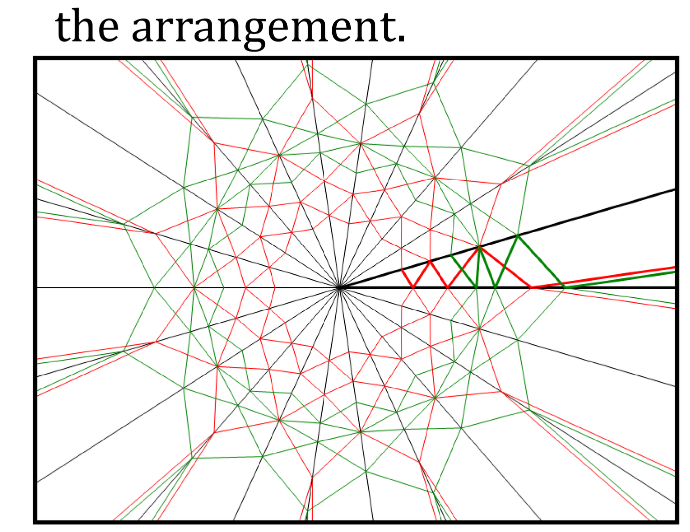
A two beam quadri-triangular arrangement is constructed by bouncing two beams within the generating wedge in such a way as to create only triangles and quadrilaterals in the enclosed regions of the arrangement.

In general, two beam quadri-triangular arrangements are not simplicial.





By changing the pattern of segment placements, we alter the number of quadrilaterals in the arrangement.



Theorem 2: In order to construct a quadri-triangular arrangement of two beams using the Kaleidoscopic beam technique, assuming a single red beam construction has been placed, only the following pattern of green segment placements are admissible

If g_{i-1} and g_{i-2} terminate at bounding mirrors, then either:

a) g_i crosses a red segment and terminates at the opposite bounding mirror or

b) g_i bounces to the opposite red vertex

If g_{i-1} terminates at a vertex then

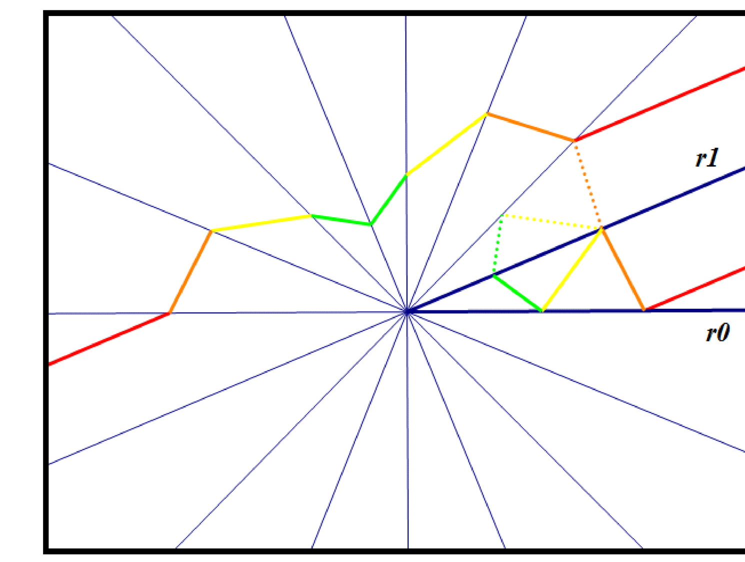
c) g_i must bounce to the opposite bounding mirror with out intersecting a red segment

If g_{i-1} terminates at a bounding mirror and g_{i-2} terminates at a red vertex then

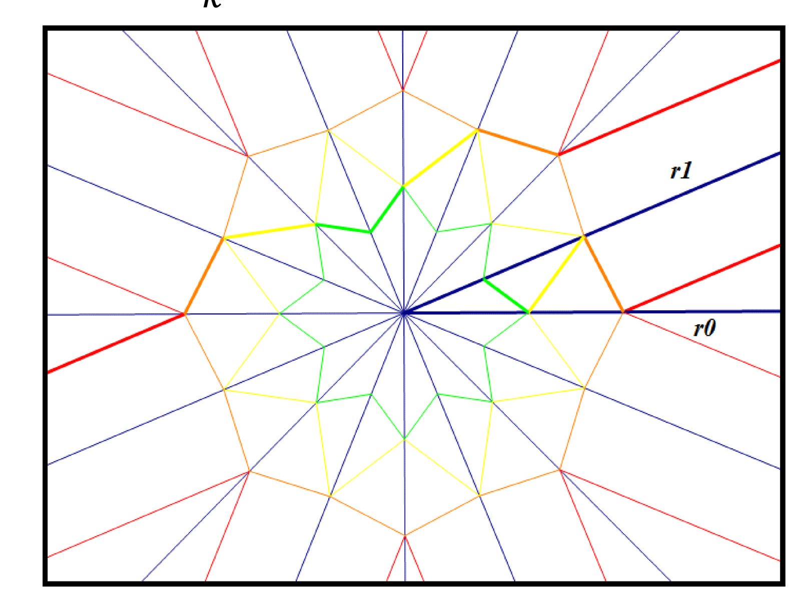
d) g_i crosses a red segment and terminates at the opposite bounding mirror

Kaleidoscopic Beam Technique

Produces (pseudo)line arrangements with the symmetries of a regular k-gon by placing beams, made up of $\left\lfloor\frac{k+1}{2}\right\rfloor$ line segments, between two rays, r_0 and r_1, which are separated by $\frac{180°}{k}$

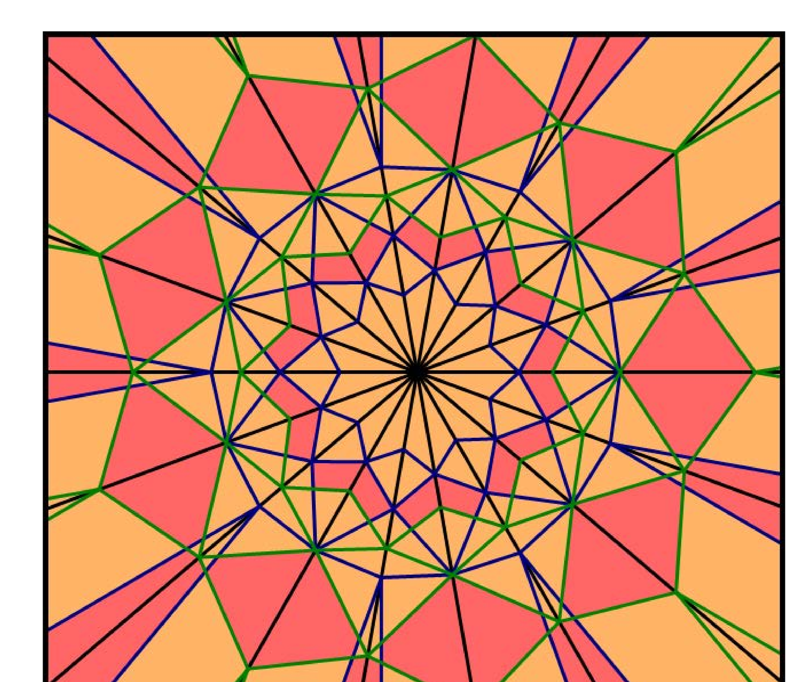


After all of the segments have been placed, this "generating wedge" which lies between r_0 and r_1 is reflected over the mirrors formed by lines which are rotations of r_0 by $\frac{180°}{k}$ about the origin.[1]



Quadri-Triangular Arrangements

Arrangements of pseudolines that partition E⁺ into regions of only triangles and quadrilaterals shall be called quadri-triangular arrangements. We shall denote the number of triangles as p_3 and the number of quadrilaterals as p_4.



References:

[1] Berman, Leah: *Symmetric Simplicial Pseudoline Arrangements*, (2008) The Electronic Journal of Combinatorics, Volume 15, Number 1.

[2] Eppstein, David: *A kaleidoscope of simplicial arrangements*, (2005) <http://11011110.livejournal.com/18849.html>

[3] Grünbaum, Branko: *A catalogue of simplicial arrangements in the real projective plane*, (2009) ARS Mathematica Contemporanea 2(2009) 1-25.

UNIVERSITY OF ALASKA FAIRBANKS